A Jump Diffusion Model For Stock Price

Gary Schurman MBE, CFA

March 2021

In Part I we built a stock price model that is a function of a Poisson process, and in Part II we built a stock price model that is a function of a Compound Poisson process. In this white paper (Part III) we will build a Jump Diffusion model for stock price.

Our Hypothetical Problem

We are tasked with building a model to forecast ABC Company stock price given the following go-forward model assumptions...

Table 1: Go-Forward Model Assumptions

Symbol	Description	Value
S_0	Stock price at time zero ($)	10.00
μ	Expected return mean (%)	15.00
σ	Expected return volatility (%)	30.00
ω	Jump size mean (%)	2.50
υ	Jump size volatility (%)	6.00
λ	Average number of annual jumps (#)	4.00
t	Time in years (#)	3.00

Our task is to answer the following questions...

Question 1: What is random stock price at the end of year 3 given that there were k = 10 jumps drawn from a Poisson distribution and y = 0.65 and x = -1.25 drawn from a normal distribution.

Question 2: What is expected conditional stock price at the end of year 3 given that there were 10 jumps over the time period $[0, 3]$?

Question 3: What is expected unconditional stock price at the end of year 3?

Conditional Stock Price

In Part II we defined the variable ϕ to be total return excluding jumps, the variable ω to be jump size mean, and the variable υ to be jump size volatility. In Part II we defined random conditional stock price via the Compensated Poisson Process to be the following equation... [2]

$$S(k)_t = S_0 \,\mathrm{Exp}\left\{\mu\, t - \lambda\,\omega\, t + k\, \ln(1+\omega) - k\,\frac{1}{2}\,\upsilon^2 + \upsilon\,\sqrt{k}\,y\right\} \text{ ...where... } \mu = \phi + \lambda\,\omega \text{ ...and... } y \sim N\Big[0,1\Big] \quad (1)$$

Using Equation (1) above the equations for expected conditional and unconditional stock price from Part II are... [2]

$$\mathbb{E}\Big[S(k)_t\Big] = S_0\,\mathrm{Exp}\Big\{\mu\,t - \lambda\,\omega\,t + k\,\ln(1+\omega)\Big\} \text{ ...and... } \mathbb{E}\Big[S_t\Big] = S_0\,\mathrm{Exp}\Big\{\mu\,t\Big\} \quad (2)$$

In Equation (2) above the variable μ is a known constant (i.e. is not random). For our jump diffusion model we want to make the variable μ a normally-distributed random variable (the diffusion part of jump diffusion). If we define the variable σ to be return volatility and the variable x to be a normally-distributed random variable with mean zero and variance one then the equation for random total return is...

$$\text{random } \mu\,t = \mu\,t - \frac{1}{2}\,\sigma^2\,t + \sigma\,\sqrt{t}\,x \text{ ...where... } x \sim N\Big[0,1\Big] \quad (3)$$

Note that the random variable x in Equation (3) above is independent of the random variable y in Equation (1) above. Using the definition in Equation (3) above we can rewrite Equation (1) above as...

$$S(k)_t = S_0 \,\mathrm{Exp}\left\{\mu\, t - \lambda\,\omega\, t + k\, \ln(1+\omega) - \frac{1}{2}\,\sigma^2\, t - k\,\frac{1}{2}\,\upsilon^2 + \sigma\,\sqrt{t}\,x + \upsilon\,\sqrt{k}\,y\right\} \tag{4}$$

We will define the random variable A to be the following equation...

$$\text{if... } A = \mathrm{Exp}\left\{\mu\, t - \frac{1}{2}\,\sigma^2\, t + \sigma\,\sqrt{t}\,x\right\} \text{ ...then... } A \sim N\left[\mu\, t - \frac{1}{2}\,\sigma^2\, t, \sigma^2\, t\right] \text{ ...because... } x \sim N\left[0,1\right] \tag{5}$$

The equation for the expected value of Equation (5) above is...

$$\mathbb{E}\left[A\right] = \mathrm{Exp}\left\{\text{mean} + \frac{1}{2}\,\text{variance}\right\} = \mathrm{Exp}\left\{\mu\, t - \frac{1}{2}\,\sigma^2\, t + \frac{1}{2}\,\sigma^2\, t\right\} = \mathrm{Exp}\left\{\mu\, t\right\} \tag{6}$$

We will define the random variable B to be the following equation...

$$\text{if... } B = \mathrm{Exp}\left\{-\,k\,\frac{1}{2}\,\upsilon^2 + \upsilon\,\sqrt{k}\,y\right\} \text{ ...then... } B \sim N\left[-\,k\,\frac{1}{2}\,\upsilon^2, k\,\upsilon^2\right] \text{ ...because... } y \sim N\left[0,1\right] \tag{7}$$

The equation for the expected value of Equation (7) above is...

$$\mathbb{E}\left[B\right] = \mathrm{Exp}\left\{\text{mean} + \frac{1}{2}\,\text{variance}\right\} = \mathrm{Exp}\left\{-\,k\,\frac{1}{2}\,\upsilon^2 + \frac{1}{2}\,k\,\upsilon^2\right\} = \mathrm{Exp}\left\{0\right\} \tag{8}$$

Using Equations (5) and (7) above we can rewrite Equation (4) above as...

$$S(k)_t = S_0 \,\mathrm{Exp}\left\{-\,\lambda\,\omega\, t + k\, \ln(1+\omega)\right\} A\, B \tag{9}$$

If the standardized normally-distributed random variables y and z above are independent, which they are, then the equation for the expected value of Equation (9) above is...

$$\mathbb{E}\left[S(k)_t\right] = S_0 \,\mathrm{Exp}\left\{-\,\lambda\,\omega\, t + k\, \ln(1+\omega)\right\} \mathbb{E}\left[A\right] \mathbb{E}\left[B\right] \text{ ...when... } \mathbb{E}\left[x\, y\right] = 0 \tag{10}$$

Using Equations (6) and (8) above the solution to Equation (10) above is...

$$\mathbb{E}\left[S(k)_t\right] = S_0 \,\mathrm{Exp}\left\{-\,\lambda\,\omega\, t + k\, \ln(1+\omega)\right\} \mathrm{Exp}\left\{\mu\, t\right\} \mathrm{Exp}\left\{0\right\} = S_0 \,\mathrm{Exp}\left\{\mu\, t - \lambda\,\omega\, t + k\, \ln(1+\omega)\right\} \tag{11}$$

Unconditional Stock Price

In Part One we defined the variable λ to be jump intensity, which is the average number of jumps realized over a given time interval, and the variable k to be the number of jumps realized over the time interval $[0,t]$. The number of jumps is a Poisson-distributed random variable. The equation for the probability of k jumps over the time interval $[0,t]$ is... [3]

$$\mathrm{Prob}\left[k\right] = \frac{(\lambda\, t)^k}{k!}\,\mathrm{Exp}\left\{-\,\lambda\, t\right\} \tag{12}$$

Using Equations (11) and (12) above the equation for expected unconditional stock price at time t is... [1]

$$\begin{aligned}
\mathbb{E}\left[S_t\right] &= \sum_{k=0}^{\infty} \frac{(\lambda\, t)^k}{k!}\,\mathrm{Exp}\left\{-\,\lambda\, t\right\} \mathbb{E}\left[S(k)_t\right] \\
&= \sum_{k=0}^{\infty} \frac{(\lambda\, t)^k}{k!}\,\mathrm{Exp}\left\{-\,\lambda\, t\right\} S_0 \,\mathrm{Exp}\left\{\mu\, t - \lambda\,\omega\, t + k\, \ln(1+\omega)\right\} \\
&= S_0 \,\mathrm{Exp}\left\{\mu\, t - \lambda\,\omega\, t\right\} \sum_{k=0}^{\infty} \frac{(\lambda\, t)^k}{k!}\,\mathrm{Exp}\left\{-\,\lambda\, t\right\} \left(1+\omega\right)^k \\
&= S_0 \,\mathrm{Exp}\left\{\mu\, t - \lambda\,\omega\, t\right\} \mathrm{Exp}\left\{\lambda\,\omega\, t\right\} \\
&= S_0 \,\mathrm{Exp}\left\{\mu\, t\right\}
\end{aligned} \tag{13}$$

The Answers To Our Hypothetical Problem

Question 1: What is random stock price at the end of year 3 given that there were k = 10 jumps drawn from a Poisson distribution and y = 0.65 and x = -1.25 drawn from a normal distribution.

Using Equation (4) above and the data in Table 1 above the answer to the question is...

$$S(10)_3 = 10.00 \times \text{Exp}\left\{0.15 \times 3 - 4 \times 0.025 \times 3 + 10 \times \ln(1 + 0.025) - \frac{1}{2} \times 0.30^2 \times 3 \right.$$
$$\left. - 10 \times \frac{1}{2} \times 0.06^2 + 0.30 \times \sqrt{3} \times -1.25 + 0.06 \times \sqrt{10} \times 0.65 \right\} = 7.54 \tag{14}$$

Question 2: What is expected conditional stock price at the end of year 3 given that there were 10 jumps over the time period $[0, 3]$?

Using Equation (11) above and the data in Table 1 above the answer to the question is...

$$\mathbb{E}\left[S(10)_3\right] = 10.00 \times \text{Exp}\left\{0.15 \times 3 - 4 \times 0.025 \times 3 + 10 \times \ln(1 + 0.025)\right\} = 14.87 \tag{15}$$

Question 3: What is expected unconditional stock price at the end of year 3?

Using Equations (13) above and the data in Table 1 above the answer to the question is...

$$\mathbb{E}\left[S_3\right] = 10.00 \times \text{Exp}\left\{0.1500 \times 3\right\} = 15.68 \tag{16}$$

References

[1] Gary Schurman, *The Poisson Process,* March, 2021.

[2] Gary Schurman, *The Compensated Poisson Process,* March, 2021.

[3] Gary Schurman, *The Poisson Distribution,* June, 2012.